

Solution The value of the integral of f over R is

$$\begin{aligned}\int_0^\pi \int_0^1 x \cos xy \, dy \, dx &= \int_0^\pi \left[\sin xy \right]_{y=0}^{y=1} dx = \int_0^\pi x \cos xy \, dy = \sin xy + C \\ &= \int_0^\pi (\sin x - 0) \, dx = -\cos x \Big|_0^\pi = 1 + 1 = 2.\end{aligned}$$

The area of R is π . The average value of f over R is $2/\pi$. ■

Exercises 15.3

Area by Double Integrals

In Exercises 1–12, sketch the region bounded by the given lines and curves. Then express the region's area as an iterated double integral and evaluate the integral.

- The coordinate axes and the line $x + y = 2$
- The lines $x = 0$, $y = 2x$, and $y = 4$
- The parabola $x = -y^2$ and the line $y = x + 2$
- The parabola $x = y - y^2$ and the line $y = -x$
- The curve $y = e^x$ and the lines $y = 0$, $x = 0$, and $x = \ln 2$
- The curves $y = \ln x$ and $y = 2 \ln x$ and the line $x = e$, in the first quadrant
- The parabolas $x = y^2$ and $x = 2y - y^2$
- The parabolas $x = y^2 - 1$ and $x = 2y^2 - 2$
- The lines $y = x$, $y = x/3$, and $y = 2$
- The lines $y = 1 - x$ and $y = 2$ and the curve $y = e^x$
- The lines $y = 2x$, $y = x/2$, and $y = 3 - x$
- The lines $y = x - 2$ and $y = -x$ and the curve $y = \sqrt{x}$

Identifying the Region of Integration

The integrals and sums of integrals in Exercises 13–18 give the areas of regions in the xy -plane. Sketch each region, label each bounding curve with its equation, and give the coordinates of the points where the curves intersect. Then find the area of the region.

- $\int_0^6 \int_{y^2/3}^{2y} dx \, dy$
- $\int_0^3 \int_{-x}^{x(2-x)} dy \, dx$
- $\int_0^{\pi/4} \int_{\sin x}^{\cos x} dy \, dx$
- $\int_{-1}^2 \int_{y^2}^{y+2} dx \, dy$
- $\int_{-1}^0 \int_{-2x}^{1-x} dy \, dx + \int_0^2 \int_{-x/2}^{1-x} dy \, dx$
- $\int_0^2 \int_{x^2-4}^0 dy \, dx + \int_0^4 \int_0^{\sqrt{x}} dy \, dx$

Finding Average Values

- Find the average value of $f(x, y) = \sin(x + y)$ over
 - the rectangle $0 \leq x \leq \pi$, $0 \leq y \leq \pi$.
 - the rectangle $0 \leq x \leq \pi$, $0 \leq y \leq \pi/2$.
- Which do you think will be larger, the average value of $f(x, y) = xy$ over the square $0 \leq x \leq 1$, $0 \leq y \leq 1$, or the

average value of f over the quarter circle $x^2 + y^2 \leq 1$ in the first quadrant? Calculate them to find out.

- Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \leq x \leq 2$, $0 \leq y \leq 2$.
- Find the average value of $f(x, y) = 1/(xy)$ over the square $\ln 2 \leq x \leq 2 \ln 2$, $\ln 2 \leq y \leq 2 \ln 2$.

Theory and Examples

- Geometric area** Find the area of the region

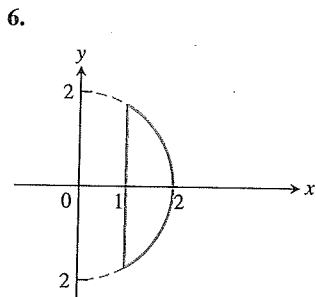
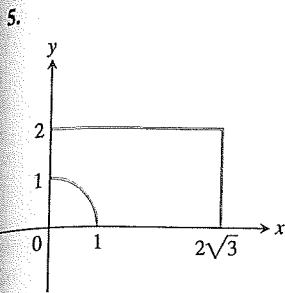
$$R: 0 \leq x \leq 2, 2 - x \leq y \leq \sqrt{4 - x^2},$$

using (a) Fubini's Theorem, (b) simple geometry.

- Geometric area** Find the area of the circular washer with outer radius 2 and inner radius 1, using (a) Fubini's Theorem, (b) simple geometry.
- Bacterium population** If $f(x, y) = (10,000e^y)/(1 + |x|/2)$ represents the "population density" of a certain bacterium on the xy -plane, where x and y are measured in centimeters, find the total population of bacteria within the rectangle $-5 \leq x \leq 5$ and $-2 \leq y \leq 0$.
- Regional population** If $f(x, y) = 100(y + 1)$ represents the population density of a planar region on Earth, where x and y are measured in kilometers, find the number of people in the region bounded by the curves $x = y^2$ and $x = 2y - y^2$.
- Average temperature in Texas** According to the *Texas Almanac*, Texas has 254 counties and a National Weather Service station in each county. Assume that at time t_0 , each of the 254 weather stations recorded the local temperature. Find a formula that would give a reasonable approximation of the average temperature in Texas at time t_0 . Your answer should involve information that you would expect to be readily available in the *Texas Almanac*.
- If $y = f(x)$ is a nonnegative continuous function over the closed interval $a \leq x \leq b$, show that the double integral definition of area for the closed plane region bounded by the graph of f , the vertical lines $x = a$ and $x = b$, and the x -axis agrees with the definition for area beneath the curve in Section 5.3.
- Suppose $f(x, y)$ is continuous over a region R in the plane and that the area $A(R)$ of the region is defined. If there are constants m and M such that $m \leq f(x, y) \leq M$ for all $(x, y) \in R$, prove that

$$mA(R) \leq \iint_R f(x, y) \, dA \leq MA(R).$$

- Suppose $f(x, y)$ is continuous and nonnegative over a region R in the plane with a defined area $A(R)$. If $\iint_R f(x, y) \, dA = 0$, prove that $f(x, y) = 0$ at every point $(x, y) \in R$.



7. The region enclosed by the circle $x^2 + y^2 = 2x$
 8. The region enclosed by the semicircle $x^2 + y^2 = 2y, y \geq 0$

Evaluating Polar Integrals

In Exercises 9–22, change the Cartesian integral into an equivalent polar integral. Then evaluate the polar integral.

9. $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} dy dx$ 10. $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2) dx dy$
 11. $\int_0^2 \int_0^{\sqrt{4-y^2}} (x^2 + y^2) dx dy$
 12. $\int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy dx$ 13. $\int_0^6 \int_0^y x dx dy$
 14. $\int_0^2 \int_0^x y dy dx$ 15. $\int_1^{\sqrt{3}} \int_1^x dy dx$
 16. $\int_{\sqrt{2}}^2 \int_{\sqrt{4-y^2}}^y dx dy$ 17. $\int_{-1}^0 \int_{-\sqrt{1-x^2}}^0 \frac{2}{1 + \sqrt{x^2 + y^2}} dy dx$
 18. $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1 + x^2 + y^2)^2} dy dx$
 19. $\int_0^{\ln 2} \int_0^{\sqrt{(\ln 2)^2 - y^2}} e^{\sqrt{x^2 + y^2}} dx dy$
 20. $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2 + y^2 + 1) dx dy$
 21. $\int_0^1 \int_x^{\sqrt{2-x^2}} (x + 2y) dy dx$
 22. $\int_1^2 \int_0^{\sqrt{2x-x^2}} \frac{1}{(x^2 + y^2)^2} dy dx$

In Exercises 23–26, sketch the region of integration and convert each polar integral or sum of integrals to a Cartesian integral or sum of integrals. Do not evaluate the integrals.

23. $\int_0^{\pi/2} \int_0^1 r^3 \sin \theta \cos \theta dr d\theta$
 24. $\int_{\pi/6}^{\pi/2} \int_1^{\csc \theta} r^2 \cos \theta dr d\theta$

25. $\int_0^{\pi/4} \int_0^{2 \sec \theta} r^5 \sin^2 \theta dr d\theta$
 26. $\int_0^{\tan^{-1} \frac{4}{3}} \int_0^{3 \sec \theta} r^7 dr d\theta + \int_{\tan^{-1} \frac{4}{3}}^{\pi/2} \int_0^{4 \csc \theta} r^7 dr d\theta$

Area in Polar Coordinates

27. Find the area of the region cut from the first quadrant by the curve $r = 2(2 - \sin 2\theta)^{1/2}$.
 28. **Cardioid overlapping a circle** Find the area of the region that lies inside the cardioid $r = 1 + \cos \theta$ and outside the circle $r = 1$.
 29. **One leaf of a rose** Find the area enclosed by one leaf of the rose $r = 12 \cos 3\theta$.
 30. **Snail shell** Find the area of the region enclosed by the positive x -axis and spiral $r = 4\theta/3, 0 \leq \theta \leq 2\pi$. The region looks like a snail shell.
 31. **Cardioid in the first quadrant** Find the area of the region cut from the first quadrant by the cardioid $r = 1 + \sin \theta$.
 32. **Overlapping cardioids** Find the area of the region common to the interiors of the cardioids $r = 1 + \cos \theta$ and $r = 1 - \cos \theta$.

Average Values

In polar coordinates, the **average value** of a function over a region R (Section 15.3) is given by

$$\frac{1}{\text{Area}(R)} \iint_R f(r, \theta) r dr d\theta.$$

33. **Average height of a hemisphere** Find the average height of the hemispherical surface $z = \sqrt{a^2 - x^2 - y^2}$ above the disk $x^2 + y^2 \leq a^2$ in the xy -plane.
 34. **Average height of a cone** Find the average height of the (single) cone $z = \sqrt{x^2 + y^2}$ above the disk $x^2 + y^2 \leq a^2$ in the xy -plane.
 35. **Average distance from interior of disk to center** Find the average distance from a point $P(x, y)$ in the disk $x^2 + y^2 \leq a^2$ to the origin.
 36. **Average distance squared from a point in a disk to a point in its boundary** Find the average value of the *square* of the distance from the point $P(x, y)$ in the disk $x^2 + y^2 \leq 1$ to the boundary point $A(1, 0)$.

Theory and Examples

37. **Converting to a polar integral** Integrate $f(x, y) = [\ln(x^2 + y^2)]/\sqrt{x^2 + y^2}$ over the region $1 \leq x^2 + y^2 \leq e$.
 38. **Converting to a polar integral** Integrate $f(x, y) = [\ln(x^2 + y^2)]/(x^2 + y^2)$ over the region $1 \leq x^2 + y^2 \leq e^2$.
 39. **Volume of noncircular right cylinder** The region that lies inside the cardioid $r = 1 + \cos \theta$ and outside the circle $r = 1$ is the base of a solid right cylinder. The top of the cylinder lies in the plane $z = x$. Find the cylinder's volume.
 40. **Volume of noncircular right cylinder** The region enclosed by the lemniscate $r^2 = 2 \cos 2\theta$ is the base of a solid right cylinder whose top is bounded by the sphere $z = \sqrt{2 - r^2}$. Find the cylinder's volume.

41. Converting to polar integrals

- a. The usual way to evaluate the improper integral $I = \int_0^\infty e^{-x^2} dx$ is first to calculate its square:

$$I^2 = \left(\int_0^\infty e^{-x^2} dx \right) \left(\int_0^\infty e^{-y^2} dy \right) = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy.$$

Evaluate the last integral using polar coordinates and solve the resulting equation for I .

- b. Evaluate

$$\lim_{x \rightarrow \infty} \operatorname{erf}(x) = \lim_{x \rightarrow \infty} \int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}} dt.$$

42. Converting to a polar integral Evaluate the integral

$$\int_0^\infty \int_0^\infty \frac{1}{(1+x^2+y^2)^2} dx dy.$$

43. Existence Integrate the function $f(x, y) = 1/(1 - x^2 - y^2)$ over the disk $x^2 + y^2 \leq 3/4$. Does the integral of $f(x, y)$ over the disk $x^2 + y^2 \leq 1$ exist? Give reasons for your answer.

44. Area formula in polar coordinates Use the double integral in polar coordinates to derive the formula

$$A = \int_\alpha^\beta \int_\alpha^{\beta} \frac{1}{2} r^2 d\theta$$

for the area of the fan-shaped region between the origin and polar curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$.

45. Average distance to a given point inside a disk Let P_0 be a point inside a circle of radius a and let h denote the distance from P_0 to the center of the circle. Let d denote the distance from an arbitrary point P to P_0 . Find the average value of d^2 over the region enclosed by the circle. (Hint: Simplify your work by placing the center of the circle at the origin and P_0 on the x -axis.)

46. Area Suppose that the area of a region in the polar coordinate plane is

$$A = \int_{\pi/4}^{3\pi/4} \int_{\csc \theta}^{2 \sin \theta} r dr d\theta.$$

Sketch the region and find its area.

47. Evaluate the integral $\iint_R \sqrt{x^2 + y^2} dA$, where R is the region inside the upper semicircle of radius 2 centered at the origin, but outside the circle $x^2 + (y - 1)^2 = 1$.
48. Evaluate the integral $\iint_R (x^2 + y^2)^{-2} dA$, where R is the region inside the circle $x^2 + y^2 = 2$ for $x \leq -1$.

COMPUTER EXPLORATIONS

In Exercises 49–52, use a CAS to change the Cartesian integrals into an equivalent polar integral and evaluate the polar integral. Perform the following steps in each exercise.

- Plot the Cartesian region of integration in the xy -plane.
- Change each boundary curve of the Cartesian region in part (a) to its polar representation by solving its Cartesian equation for r and θ .
- Using the results in part (b), plot the polar region of integration in the $r\theta$ -plane.
- Change the integrand from Cartesian to polar coordinates. Determine the limits of integration from your plot in part (c) and evaluate the polar integral using the CAS integration utility.

49. $\int_0^1 \int_x^1 \frac{y}{x^2 + y^2} dy dx$

50. $\int_0^1 \int_0^{x/2} \frac{x}{x^2 + y^2} dy dx$

51. $\int_0^1 \int_{-y/3}^{y/3} \frac{y}{\sqrt{x^2 + y^2}} dx dy$

52. $\int_0^1 \int_y^{2-y} \sqrt{x + y} dx dy$

15.5 Triple Integrals in Rectangular Coordinates

Just as double integrals allow us to deal with more general situations than could be handled by single integrals, triple integrals enable us to solve still more general problems. We use triple integrals to calculate the volumes of three-dimensional shapes and the average value of a function over a three-dimensional region. Triple integrals also arise in the study of vector fields and fluid flow in three dimensions, as we will see in Chapter 16.

Triple Integrals

If $F(x, y, z)$ is a function defined on a closed bounded region D in space, such as the region occupied by a solid ball or a lump of clay, then the integral of F over D may be defined in the following way. We partition a rectangular boxlike region containing D into rectangular cells by planes parallel to the coordinate axes (Figure 15.30). We number the cells that lie completely inside D from 1 to n in some order, the k th cell having dimensions Δx_k by Δy_k by Δz_k and volume $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$. We choose a point (x_k, y_k, z_k) in each cell and form the sum

$$S_n = \sum_{k=1}^n F(x_k, y_k, z_k) \Delta V_k. \tag{1}$$

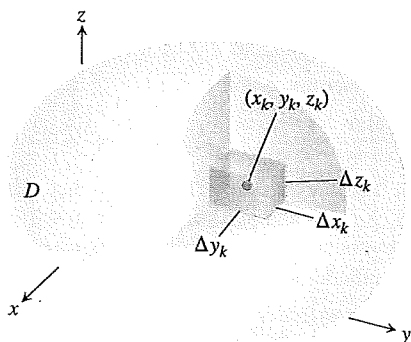


FIGURE 15.30 Partitioning a solid with rectangular cells of volume ΔV_k .